Determination of Critical Slip Surfaces Using Mutative Scale Chaos Optimization

Cong Hu ; Rafael Jimenez, Ph.D. , Shu-cai Li, Ph.D. , and Li-ping Li, Ph.D.

Abstract: Limit equilibrium is a common method used to analyze the stability of a slope, and minimization of the factor of safety or identification of critical slip surfaces is a classical geotechnical problem in the context of limit equilibrium methods for slope stability analyses. A mutative scale chaos optimization algorithm is employed in this study to locate the noncircular critical slip surface with Spencer's method being employed to compute the factor of safety. Four examples from the literature—one homogeneous slope and three layered slopes—are employed to identify the efficiency and accuracy of this approach. Results indicate that the algorithm is flexible and that although it does not generally provide the minimum FS, it provides results that are close to the minimum, an improvement over other solutions proposed in the literature and with small relative errors with respect to other minimum factor of safety (FS) values reported in the literature.

Author keywords: Mutative scale chaos optimization; Critical slip surface; Slope stability; Factor of safety.

Introduction

Slope stability is a key aspect of geotechnical engineering. The economic consequences of landslides are significant, as they can impose hazards on human life and property (Schuster 1996). There are several methods for slope stability analysis: the most popular are limit equilibrium and finite elements (FEM) (Griffiths and Lane 1999; Zhang 1999). For instance, Zou et al. (1995) and Zheng et al. (2009a) developed new finite element procedures to search the critical slip surface in two-dimensional slope stability analyses. But FE analyses require more advanced input information, such as initial and boundary conditions, stress-strain properties and loading sequences (Tan and Sarma 2008). Thus, given its simplicity, the limit equilibrium method is still, and will probably continue to be, widely used by designers and analysts (Sarma and Tan 2006). Traditional limit equilibrium methods that fulfill force and moment equilibrium (Bishop 1955; Morgenstern and Price 1965; Spencer 1967) are the most common, although modified methods have also been employed (Baker and Garber 1978; Li and White 1987; Sarma and Tan 2006).

When limit equilibrium methods are employed to assess the stability of a slope, the identification of the critical slip surface remains. Techniques such as a *grid search* can be used for simple cases, but more advanced optimization methods are needed in many real cases. For instance, although traditional (deterministic) optimization techniques—such as simplex, steepest descent, and conjugate gradient—were tested first [Nguyen (1985); Chen and Shao (1988); Arai and Tagyo (1985)]. It was quickly found that they could have problems due to their convergence to local minima. To avoid such problems, stochastic or heuristic methods were employed as an alternative: Greco (1996) and Malkawi et al. (2001), employed Monte Carlo methods to analyze slope stability, whereas Zolfaghari et al. (2005) used a simple binary genetic algorithm (GA) for such a task. Other real-coded GAs have also been proposed; for instance, Li et al. (2010) and Jurado-Piña and Jimenez (2014). Similarly, other researchers have tested other optimization methods—such as ant-colony optimization, simulated annealing, particle swarm optimization, harmony search, and tabu search—with different success [Kahatadeniya et al. (2009); Cheng (2003); Cheng et al. (2007a); Cheng et al. (2007b)].

In this work, such efforts are continued to explore a new optimization technique to identify the critical geometry: the mutative scale chaos optimization algorithm.

Limit Equilibrium Method for Slope Stability

Spencer's method (1967) is employed to compute the factor of safety (FS). Spencer's method is a general limit equilibrium method that satisfies equilibrium of forces and moments. The solution is obtained in terms of the FS—it is the factor by which strength parameters must be divided to achieve a limit equilibrium conditions—and the (constant) angle of inclination of interslice forces, θ. A detailed formulation of the method can be found in Spencer (1967) or in Duncan and Wright (2005) and, for the sake of brevity, will not be repeated herein. While Zheng et al. (2009b) found that in some cases convergence cannot be obtained and more robust solution procedures should be tried for Spencer's method.

To conduct the computations, the implementation of Spencer's method available in the program Slope8R (Duncan and Song 1984) is employed, although minor modifications are implemented on the source code to improve convergence.

Optimization to Identify Critical Slip Surfaces

Description of Chaos Optimization

Chaos theory—or the *butterfly effect*—was originally proposed by Lorenz (1963). It is a global optimization method with three main characteristics: pseudo-randomness, ergodicity, and irregularity. For a general minimization problem, chaos optimization involves several steps (Yang et al. 2007): (1) generate a sequence of chaotic points, mapping them to design points in the original space of design variables and compute the objective function for such design points, choosing the point with the minimum function value as the current optimum; and (2) iterate to obtain the global optimum.

The logistic map of chaos variables can be defined as

$$z^{k+1} = \mu z^k(1 - z^k)(k = 0, 1, 2, 3, \ldots,) \tag{1}$$

where z is a chaos variable and μ is a control parameter (if $\mu = 4$, as used in this case, very small changes of the initial z will produce large differences in the long-term behavior).

In general, an optimization problem with a nonlinear, and probably multimodal, function with boundary constraints can be formulated as

$$\text{Minimize } f(x) = f(x_1, x_2, x_3 \ldots x_m)$$

$$\text{Subject to } \mathrm{XL}_i \le x_i \le \mathrm{XU}_i \qquad i = 1, 2, 3 \ldots\ldots$$

where f is the objective function; $x_i (i = 1, 2, \ldots n)$ are design or *optimization* variables; and XL_i and XU_i are their lower and upper bounds.

To transform chaos variables into real (physical) variables, the following linear map can be employed:

$$z_i = (x_i - \mathrm{XL}_i)/(\mathrm{XU}_i - \mathrm{XL}_i) \tag{2}$$

Then, chaos optimization can be conducted as (Yang et al. 2007)

1. Initialize the chaos variables z^0. Calculate initial design variables $x^0 = \mathrm{XL}_i + z^0(\mathrm{XU}_i - \mathrm{XL}_i)$ and the objective function $f^* = f(x^0)$.
2. Compute new chaos variables by Eq. (1) and the new design variables by the linear mapping $x^{k+1} = \mathrm{XL}_i + z^{k+1}(\mathrm{XU}_i - \mathrm{XL}_i)$. Evaluate the objective function at x^{k+1}
3. Check the convergence criterion: If it is satisfied, stop; otherwise, return to Step 2.

Zhang et al. (1999) found that the method described above is only effective with a small design space; hence, an alternative mutative scale chaos optimization algorithm was presented to reduce runtime and to improve the accuracy (Zhang 1999; Xu et al. 2013). Such algorithm is described in Appendix I.

Mutative Scale Chaos Optimization for Slope Stability

The mutative scale chaos optimization algorithm described in Appendix I is employed to identify critical slip surfaces. To produce a mapping with uniform probability density, the Kent map (Yang et al. 2007) is employed in this work. The Kent map is

$$f(z)\begin{cases} z/\delta & 0 < z \le \delta \\ (1 - z)/(1 - \delta) & \delta < z \le 1 \end{cases} \tag{3}$$

where parameter δ is set to a value within a (0,1) interval. ($\delta = 0.3$ is assumed herein.)

A slip surface with n vertices can be represented by the coordinates of its vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Although some authors have considered slip surfaces with *kinks* (Cheng et al. 2007a), in most cases it is reasonable to consider upwards-concave slip surfaces; that is,

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \tag{4}$$

where α_i is the angle formed by the base of a slice with the horizontal plane (Fig. 1).

To facilitate the description of the algorithm, imagine that slip surfaces with Nnodes = 8 vertices. To define the slip surface, x coordinates of the *extreme* vertices $N1$ and $N8$ are initially generated. To that end, chaos is used to generate the x coordinates within the *allowable extreme intervals* given by ($N_{1\,\min}$, $N_{1\,\max}$) and ($N_{8\,\min}$, $N_{8\,\max}$). Once the x coordinates of extreme vertices are

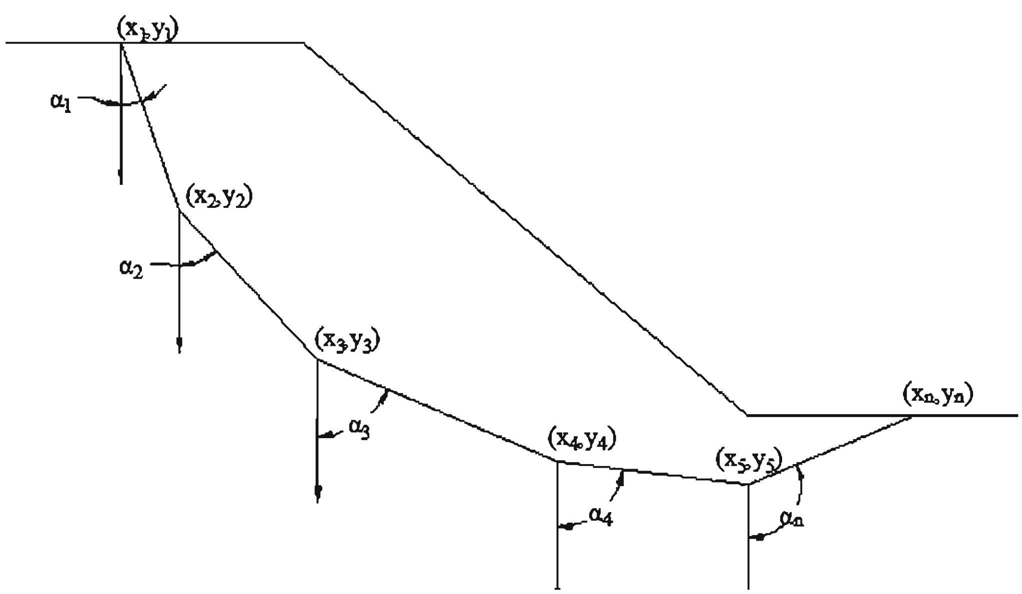

Fig. 1. Acceptable slip surface

known, their corresponding y coordinates can be obtained directly from the slope's surface geometry. Then, the lower and upper search bounds are defined for interior nodes of the slip surface, V_i ($i = 1, \ldots, N$nodes-2). The procedure is as follows (Fig. 2):

1. Once the locations of end vertices, N_1 and N_8, are given, the N_1N_8 segment is divided into Nnodes-1 equal-sized intervals; the coordinates of their interior extremes are called $V_1, V_1, \ldots, V_6$
2. Lines perpendicular to N_1N_8 passing through such interval extremes are defined as $V_1V_1', \ldots, V_6V_6'$; the lengths of $V_1V_1', \ldots, V_6V_6'$ are selected so that the V_i' points lie beyond the locations of expected slip surfaces.
3. Values of limit angles, β_1, β_2, γ_1, γ_2 are assumed based on experience and the soil's mechanics (for instance, Rankine's criteria can be employed as reference).
4. The intersections of $V_{1\,\min}, V_{2\,\min}, \ldots V_{6\,\min}$ and $V_{1\,\max}, V_{2\,\max}, \ldots V_{6\,\max}$ are computed. Then the lengths of $V_1V_{1\,\min}, \ldots, V_6V_{6\,\min}$ and $V_1V_{1\,\max}, \ldots, V_6V_{6\,\max}$ are obtained; respectively, they represent the lower and upper bounds for the design variables.
5. The lengths of the $V_1N_2, \ldots, V_6N_7$ segments are the design variables. Once they are defined, the coordinates of the interior nodes, N_2 to N_7, are obtained using simple geometry.

Following steps (1)–(5) above, an initial slip surface $(N_1, N_2, \ldots, N_8)$ is generated by chaos. (If not kinematically acceptable, the process is repeated until an acceptable slip surface is obtained.) Then, the FS for such slip surface is computed using Spencer's method, and FS is compared with the current minimum factor of safety, FS*. If FS is smaller than FS*, it replaces it; otherwise, a new slip surface is generated by chaos.

This procedure is repeated until convergence; at that moment, the minimum factor of safety is taken as the current one. Then the extreme vertices are regenerated by chaos, repeating the process to obtain the new (minimum) FS. Such new minimum FS is compared with the previous (recorded) one and the smaller one is saved. This process is repeated several times, and the smallest factor of safety obtained is recorded as the current factor of safety.

The coordinates of $N1$ and $N8$ corresponding to the current factor of safety are fixed.

Then the new lengths of $V_1V_{1\,\min}, \ldots, V_6V_{6\,\min}$ and $V_1V_{1\,\max}, \ldots, V_6V_{6\,\max}$ are obtained according to the equations in step 7 of Appendix I. Then, steps 7 to 11 of Appendix I are executed. The smallest obtained factor of safety is recorded as the new current factor of safety. In this step the design space is compressed.

The next step is to increase the slip surface to Nnodes = 15 vertices. To that end, midpoints are inserted in each segment of the lower search bounds $(N_1V_{1\,\min}, \ldots, V_{6\,\min}N_8)$ and upper search bounds $(N_1V_{1\,\max}, \ldots V_{6\,\max}N_8)$. Then steps (1)–(5) above are repeated several times. The factor of safety obtained in this step is the final current factor of safety.

Application Examples

To evaluate the applicability of mutative scale chaos optimization, four benchmark examples from the literature that include slopes

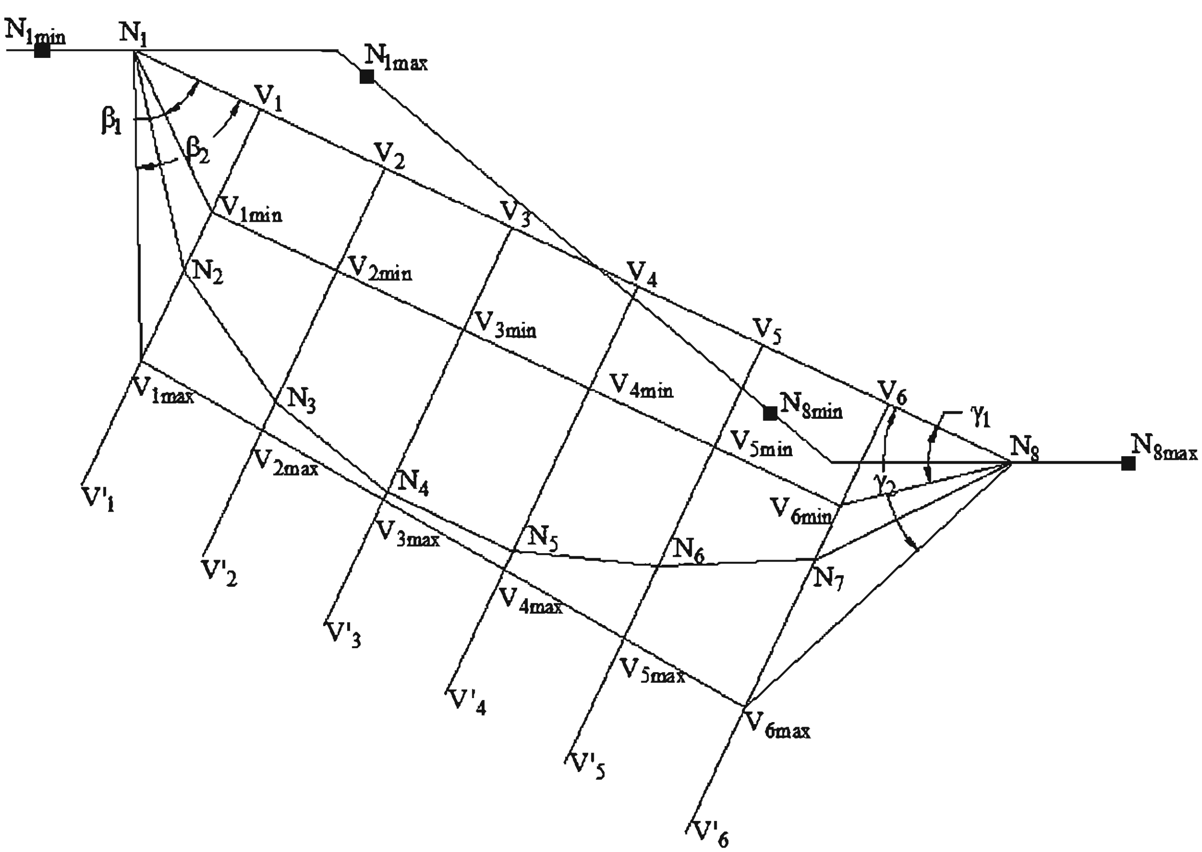

Fig. 2. Encoding of the slip surface and search bounds for vertices

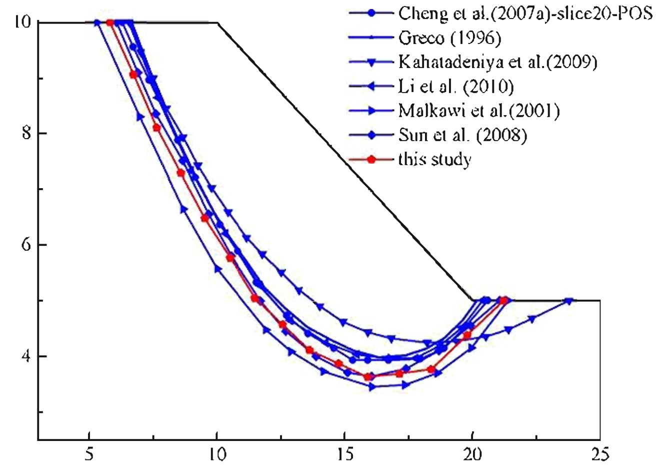

Fig. 3. Cross section and critical slip surface for example 1

Table 1. Geotechnical Properties of Soil Layers

Problem	Layer of soil	Cohesion c (kPa)	Friction φ (°)	Density γ (kN/m^3)
Example 1	1	9.8	10	17.64
Example 2	1	15	20	19
	2	17	21	19
	3	5	10	19
	4	35	28	19
Example 3	1	15	20	19
	2	17	21	19
	3	5	10	19
	4	35	28	19
Example 4	1	49	29	20.38
	2	0	30	17.64
	3	7.84	20	20.38
	4	0	30	17.64

with homogeneous and inhomogeneous soils are employed. The first example corresponds to a homogeneous slope; the others are slopes with layers. The factor of safety is computed using Spencer's method, and the mutative scale chaos optimization algorithm is implemented in *MATLAB*. The coordinates of the 15 vertices corresponding to the critical slip surfaces obtained for each example case are listed in Appendix II.

Example 1

The first example is a slope in a homogeneous soil formation that was initially proposed by Yamagami and Ueta (1988). The cross section and critical slip surface are shown in Fig. 3, whereas soil properties are listed in Table 1. Fig. 3 also shows the critical slip surface evaluated by other researchers. Table 2 reports the computed minimum FS and compares it with other results in the literature. It cannot be found that the minimum factor of safety of this example is 1.238, which was obtained by Malkawi et al. (2001). The FS reported by most researchers for this case is usually between 1.32 and 1.34, whereas the mutative chaos technique produces a slightly higher value of 1.3621. (The minimum FS computed by chaos is 1.3706 when the number of slices is seven.) The evolution of the computed minimum FS during *convergence* of the algorithm can also be presented. Fig. 4 presents the evolution of minimum FS as a function of the search cycles of example one. (*Search cycles* represent the main loop of the search algorithm, and each loop includes the subcycles described above that allow finding a minimum FS for the main loop.) In the initial search interval, the minimum factor of safety fluctuated greatly and the minimum value of each search cycle is not small enough. However, as the search interval deceases and the number of vertices increase, the minimum factor of safety of each cycle decreases gradually.

Example 2

Example 2 is originally presented by Zolfaghari et al. (2005). The geological profile is shown in Fig. 5 and the soil properties are listed in Table 1. Fig. 5 also shows the critical slip surface

Table 2. Minimum Factors of Safety for Example 1

References	Search method[a]	Limit equilibrium method	Minimum factor of safety
Yamagami and Ueta (1988)	BFGS	Spencer	1.338
	DFP		1.338
	Powell		1.338
	Simplex		1.339–1.348
Greco (1996)	Monte Carlo	Spencer	1.327–1.333
	Pattern search		1.326–1.330
Malkawi et al. (2001)	Monte Carlo	Ordinary method of slice	1.238[b]
Solati and Habibagahi (2006)	GA	Janbu	1.380
Sun et al. (2008)	GA with spline	Spencer	1.321[c]
Kahatadeniya et al. (2009)	ACO	Spencer	1.311[d]
Cheng et al. (2007a)[e]	PSO	Spencer	1.3249, 1.3285, 1.3261
	MPSO		1.3273, 1.3264, 1.3259
Li et al. (2010)[f]	Real-coded GA	Spencer	1.327, 1.327, 1.329
This study	Chaos optimization	Spencer	1.3621

[a]Key: BFGS = Broyden-Fletcher-Goldfarb-Shanno; DFP = Davidson-Fletcher-Powell; GA = genetic algorithm; ACO = Ant colony optimization; PSO = Particle swarm optimization; MPSO = Modified particle swarm optimization.
[b]Probably nonphysical solution. Using slide obtained FS = 1.252 for their failure surface (approximated; as scanned from their figures).
[c]Probably nonphysical solution. Using slide obtained FS = 1.335 for their failure surface (approximated; as scanned from their figures).
[d]Probably nonphysical solution. Using slide obtained FS = 1.492 for their failure surface (approximated; as scanned from their figures).
[e]Results for $N = \{15, 20, 30\}$ nodes, respectively.
[f]Results for three analyses with different random seeds.

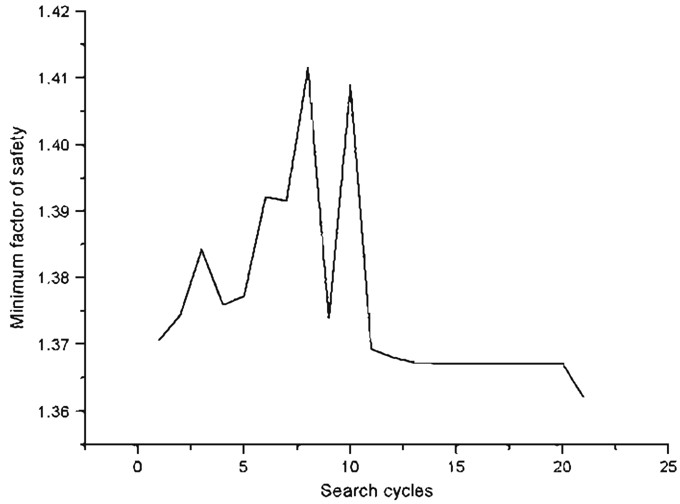

Fig. 4. Minimum factor of safety with respect to the search cycles for example 1

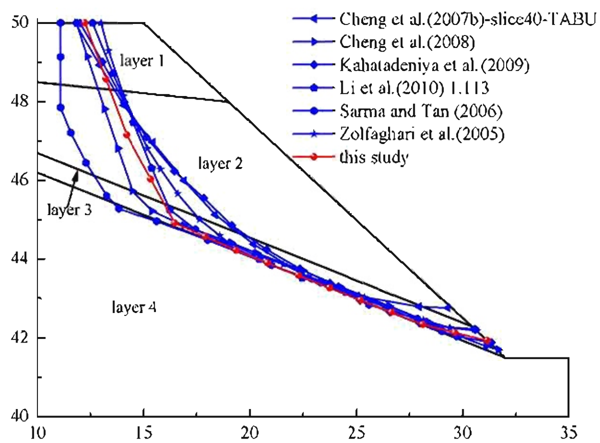

Fig. 5. Cross section and critical slip surface for example 2

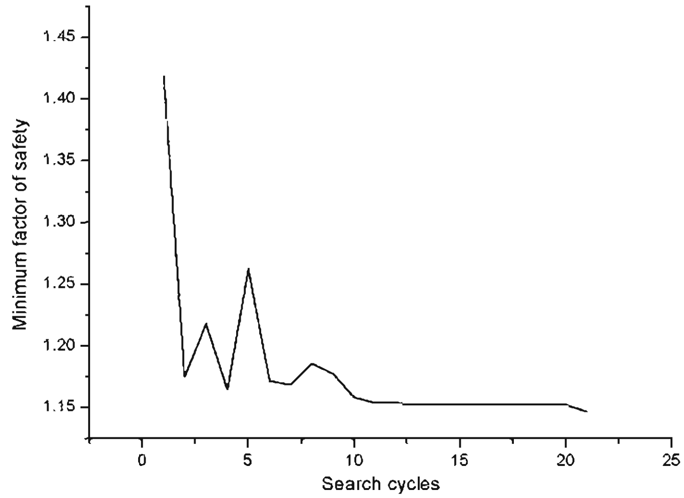

Fig. 6. Minimum factor of safety with respect to the search cycles for example 2

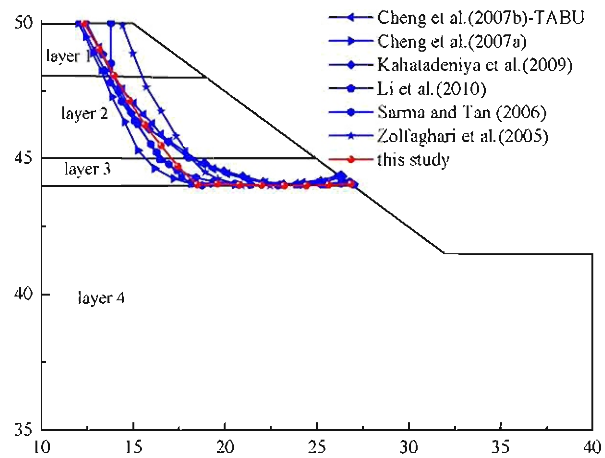

Fig. 7. Cross section and critical slip surface for example 3

Table 3. Minimum Factors of Safety for Example 2

References	Search method[a]	Limit equilibrium method	Minimum factor of safety
Zolfaghari et al. (2005)	Simple GA	Morgenstern-Price	1.24
Sarma and Tan (2006)	Critical acceleration	Sarma	1.091
Cheng et al. (2007a)[b]	PSO	Spencer	1.1055, 1.1095, 1.1010
	MPSO		1.1139, 1.1174, 1.1289
Cheng et al. (2007b)[c]	SA	Spencer	1.1789, 1.2564, 1.2813
	GA	Spencer	1.1495, 1.1117, 1.1440
	PSO	Spencer	1.1080, 1.1341, 1.1095
	SHM	Spencer	1.2512, 1.2405, 1.2068
	MHM	Spencer	1.1509, 1.1315, 1.1385
	Tabu	Spencer	1.4714, 1.4661, 1.4650
	ACO	Spencer	1.4249, 1.5299, 1.5817
Cheng et al. (2008)	AFSA	Spencer	1.1195–1.1828[d]
Kahatadeniya et al. (2009)	ACO	Spencer	1.361
Li et al. (2010)[e]	Real-coded GA	Morgenstern-Price	1.113, 1.115, 1.117
	Real-coded GA	Spencer	1.114, 1.116, 1.119
This study	Chaos optimization	Spencer	1.1471

[a]Key: GA = Genetic algorithm; PSO = Particle swarm optimization; MPSO = Modified particle swarm optimization; SA = Simulated annealing; SHM = Simple harmony search; MHM = Modified harmony search; ACO = Ant colony optimization; AFSA = Artificial fish swarms algorithm.

[b]Results for $N = \{15, 20, 30\}$ nodes, respectively.

[c]Results for $N = \{21, 31, 41\}$ nodes, respectively.

[d]The authors report values between 1.1195 and 1.1828 for 9 different cases considered.

[e]Results for three analyses with different random seeds.

Table 4. Minimum Factors of Safety for Example 3

References	Search method[a]	Limit equilibrium method	Minimum factor of safety
Zolfaghari et al. (2005)	Simple GA	Morgenstern-Price	1.48
Sarma and Tan (2006)	Critical acceleration	Sarma	1.307
Cheng et al. (2007a)[b]	PSO	Spencer	1.3323
	MPSO		1.3490
Cheng et al. (2007b)[c]	SA	Spencer	1.3961
	GA	Spencer	1.3733
	PSO	Spencer	1.3372
	SHM	Spencer	1.3729
	MHM	Spencer	1.3501
	Tabu	Spencer	1.4802
	Ant-colony	Spencer	1.5749
Kahatadeniya et al. (2009)	ACO	Spencer	1.501[d]
Li et al. (2010)[e]	Real-coded GA	Morgenstern-Price	1.335, 1.338, 1.339
	Real-coded GA	Spencer	1.336, 1.337, 1.337
This study	Chaos optimization	Spencer	1.3792

[a]Key: ACO = Ant colony optimization; GA = Genetic algorithm; MHM = Modified harmony search; MPSO = Modified particle swarm optimization; PSO = Particle swarm optimization; SA = Simulated annealing; SHM = Simple harmony search.
[b]No water pressure and no earthquake loading (for $N = 41$ nodes).
[c]No water pressure and no earthquake loading (for $N = 41$ nodes).
[d]Probably nonphysical solution. Using slide obtained FS = 1.548 for their failure surface (approximated; as scanned from their figures).
[e]Results for three analyses with different random seeds.

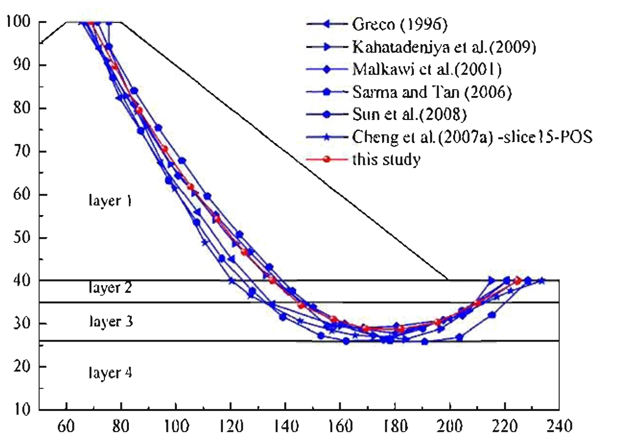

Fig. 8. Cross section and critical slip surface for example 4

evaluated by other researchers. The minimum factors of safety obtained by other researchers are listed in Table 3. The minimum factor of safety of 1.091 is obtained by Sarma and Tan (2006); although their FS value is lower, they use a different method yielding a different critical slip surface. There is also a tension crack (about 2.2 m deep) that might explain the lower FS value and the different critical slip surface (Li et al. 2010). On the other side, FS values obtained by Zolfaghari et al. (2005) and Kahatadeniya et al. (2009), as well as some of the methods tested by Cheng et al. (2007b) are larger than the result obtained in this study. The shape of their critical slip surfaces is also different, with a smaller portion of their slip surface being located within the weak layer. The minimum FS obtained with the chaos optimization approach is 1.1471, which is only about 3% larger than that obtained by Li et al. (2010). (The minimum FS with seven slices is 1.1584.) The evolution of the minimum FS with respect the search cycles is shown in Fig. 6. The trend is similar with Example 1. In the initial search interval the minimum factor of safety fluctuated greatly and the minimum value of each search cycle is not small enough. As the search interval decreases and the number of vertices increase, the minimum factor of safety of each cycle decreases gradually.

Example 3

The third example was also presented by Zolfaghari et al. (2005). (The case is studied without water pressure and without earthquake loading.) The slope consists of four layers (Fig. 7). Except for a slight inclination between the first and second layers, the other layer boundaries are horizontal. The soil properties are the same as in Example 2 (Table 1). Fig. 7 also shows the critical slip surface evaluated by other researchers. Table 4 compares the result from this study with others reported in the literature. Compared with the solutions of Zolfaghari et al. (2005), the method used in this study places more portions of the critical slip surface within the weakest layer, which explains the improved FS estimates (Cheng et al. 2007a). Besides, it was found that Kahatadeniya et al. (2009) provided an incorrect solution. However, the minimum FS results—1.3959 with seven slices and 1.3792 with 14 slices—are slightly larger than FS results of Cheng et al. (2007a) and Li et al. (2010). The evolution of the minimum FS with the search cycles presents a similar behavior and, for that reason, is not reproduced herein.

Example 4

Example 4 is also taken from Yamagami and Ueta (1988). Fig. 8 shows the geometric profile and the critical slip surface obtained in this study. The results obtained by other researchers are also shown in Fig. 8. The soil properties are also listed in Table 1. In addition to Yamagami and Ueta (1988), several other researchers have solved this problem; their results, as well as the methods employed, are listed in Table 5. The minimum factor of safety for this example, obtained by Solati and Habibagahi (2006), is 1.380, although they use Janbu's method instead of Spencer's method or some other method that fulfills both force and moment equilibrium. It was found that Malkalwi et al. (2001), Sun et al. (2008), and Kahatadeniya et al. (2009) provided incorrect solutions. The FS computed with chaos is similar to that obtained by Yamagami

Table 5. Minimum Factors of Safety for Example 4

References	Search method[a]	Limit equilibrium method	Minimum factor of safety
Yamagami and Ueta (1988)	BFGS	Spencer	1.423
	DFP		1.453
	Powell		1.402
	Simplex		1.405
Greco (1996)[b]	Monte Carlo	Spencer	1.400–1.413
	Pattern search		1.400–1.406
Malkawi et al. (2001)	Monte Carlo	Ordinary method of slice	1.333[c]
Solati and Habibagahi (2006)	GA	Janbu	1.380
Sarma and Tan (2006)	Critical acceleration	Sarma	1.422
Sun et al. (2008)	GA with spline	Spencer	1.395[d]
Kahatadeniya et al. (2009)	ACO	Spencer	1.348[e]
Cheng et al. (2007a)[f]	MPSO	Spencer (15 slices)	1.4017, 1.3967, 1.4046
	PSO		1.3862, 1.3942, 1.3983
This study	Chaos optimization	Spencer	1.4142

[a]Key: ACO = Ant colony optimization; BFGS = Broyden-Fletcher-Goldfarb-Shanno; DFP = Davidson-Fletcher-Powell; GA = Genetic algorithm; MPSO = Modified particle swarm optimization; PSO = Particle swarm optimization.
[b]Results for $N = 13$ nodes.
[c]Cheng et al. (2007a) report FS = 1.39 for their best slip surface.
[d]Solution defined by four nodal points connected by splines. Using slide, FS = 1.437 is obtained for their failure surface (approximated; as scanned from their figures).
[e]Probably nonphysical solution. Using slide, FS = 1.482 is obtained for their failure surface (approximated; as scanned from their figures).
[f]For $N = \{15, 20, 30\}$ nodes, respectively.

and Ueta (1988), Greco (1996), Sarma and Tan (2006), and Cheng et al. (2007a): a minimum FS of 1.442 with seven slices is obtained, decreasing to 1.4142 with 14 slices. The evolution of the minimum FS with the search cycles is also similar and not reproduced herein.

Conclusions

Minimization of the factor of safety, or identification of critical slip surfaces, is a classical geotechnical problem within the context of limit equilibrium methods for slope stability analyses. The problem can be formulated as an optimization problem, and different optimization approaches have been proposed in the literature.

In this paper, a novel approach is tested: the mutative scale chaos optimization algorithm. The reason for the authors' interest is that, despite recent successful applications to a different geotechnical problem (Xu et al. 2013), the algorithm has not yet been investigated for its use in slope stability. A possible encoding to implement the algorithm for slope stability problems is presented, and the implementation of the algorithm using four traditional benchmark examples from the literature is tested.

Results show that, even though the algorithm does not generally provide the minimum FS, it provides results that are close to the minimum, with small relative errors with respect to other minimum FS values reported in the literature. It is also flexible in the sense that relatively good solutions are obtained for all examples considered—which include both homogeneous and inhomogeneous slopes, as well as constrained (or mainly straight) and nonconstrained (or mainly curved) slip surfaces. In addition, the results show that the algorithm heavily depends on the number of slices considered; although convergence is faster with a lower number of slices, the minimum FS decreases about 1% to 2% when the number of slices increases from 7 to 14.

Appendix I. Mutative Scale Chaos Optimization Algorithm

The mutative scale chaos optimization (Zhang 1999 and Xu et al. 2013) can be defined as

1. Set $k = 0$, $r = 0$, $f^* = \infty$ (k represents chaos iterations, r represents the mutative scale, and f^* is the optimum value of the objective function);
2. Randomly initialize chaos variables $0 < \gamma_i^{(k)} < 1$ ($i = 1, 2, \ldots, n$). Also, initialize $a_i^{(r)} = a_i$, $b_i^{(r)} = b_i$. $a_i^{(r)}$ and $b_i^{(r)}$ denote the lower and upper boundaries, respectively;
3. Chaos variables $\gamma_i^{(k)}$ ($I = 1, 2, 3, \ldots n$) are mapped into the physical space of optimization variables
$$x_i^{(k)} = a_i^{(r)} + \gamma_i^{(k)}(b_i^{(r)} - a_i^{(r)})(i = 1, 2, \ldots n)$$
4. Compute the value of the objective function, f. If $f < f^*$, assign the value of f to f^*, the value of $\gamma_i^{(k)}$ to γ_i^*, and the value of $x_i^{(k)}$ to x_i^*. (γ_i^* is the current optimum chaos variable and x_i^* the current optimum physical variable);
5. Compute new chaos variables as
$$\gamma_i^{(k+1)} = 4\gamma_i^{(k)}(1 - \gamma_i^{(k)})$$
6. Repeat steps (3), (4), and (5) until f^* remains almost unchanged;
7. Reduce search intervals as
$$a_i^{(r+1)} = x_i^* - \lambda(b_i^{(r)} - a_i^{(r)}) \qquad b_i^{(r+1)} = x_i^* + \lambda(b_i^{(r)} - a_i^{(r)})$$
with $0 < \lambda < 0.5$. ($\lambda = 0.3$ is used herein.) If $a_i^{(r+1)} < a_i^{(r)}$, the value of $a_i^{(r)}$ is assigned to $a_i^{(r+1)}$; if $b_i^{(r+1)} > b_i^{(r)}$, the value of $b_i^{(r)}$ is assigned to $b_i^{(r+1)}$.
In addition, compute a new γ_i^* as
$$\gamma_i^* = \frac{x_i^* - a_i^{(r+1)}}{b_i^{(r+1)} - a_i^{(r+1)}}$$
8. Compute the new chaos variables as
$$v_i^{(k)} = (1 - \eta)\gamma_i^* + \eta\gamma_i^{(k)}$$
where η is a real number between 0 and 1. ($\eta = 0.2$ is used herein);
9. Repeat steps (3)–(5) using new chaos variables $v_i^{(k)}$;

10. Repeat steps (8) and (9) until f^* remains almost unchanged;
11. Repeat steps (7)–(10) until f^* remains almost unchanged, after which the algorithm stops. (The number of the repetitions usually is 4 or 5.) x_i^* is now the optimization solution and f^* is the optimum value of the objective function.

Appendix II. Coordinates for Critical Slip Surfaces

Coordinates for the Critical Slip Surfaces of Examples 1 to 4 from Uphill to Downhill

Example 1	Example 2	Example 3	Example 4
(5.81, 10.00)	(12.25, 50.00)	(12.35, 50.00)	(68.96, 100.00)
(6.72, 9.06)	(13.23, 48.58)	(13.17, 49.04)	(77.71, 89.78)
(7.63, 8.11)	(14.22, 47.16)	(13.99, 48.08)	(86.57, 79.57)
(8.57, 7.30)	(15.34, 46.04)	(14.82, 47.16)	(95.92, 70.65)
(9.53, 6.48)	(16.46, 44.92)	(15.65, 46.24)	(105.30, 61.81)
(10.51, 5.77)	(17.90, 44.57)	(16.55, 45.48)	(115.16, 54.22)
(11.49, 5.05)	(19.35, 44.22)	(17.45, 44.72)	(125.04, 46.66)
(12.55, 4.58)	(20.81, 43.90)	(18.53, 44.04)	(135.31, 40.14)
(13.62, 4.11)	(22.27, 43.58)	(19.61, 44.08)	(145.85, 34.30)
(14.75, 3.87)	(23.73, 43.27)	(20.80, 44.04)	(157.41, 31.12)
(15.89, 3.63)	(25.19, 42.95)	(22.00, 44.02)	(169.34, 28.92)
(17.13, 3.69)	(26.65, 42.64)	(23.21, 44.00)	(182.09, 28.82)
(18.37, 3.77)	(28.12, 42.33)	(24.42, 44.01)	(195.48, 30.39)
(19.78, 4.38)	(29.63, 42.13)	(25.64, 44.04)	(210.02, 34.95)
(21.20, 5.00)	(31.14, 41.93)	(26.86, 44.07)	(224.74, 40.00)

Acknowledgments

This research was funded, in part, by the State Key Development Program for Basic Research of China (No. 2013CB036000), the State Key Program of National Natural Science of China (No. 51139004), and the Independent Innovation Foundation of Shandong University (No. 2012TS064). The authors also express appreciation to the editors and reviewers for their valuable comments and suggestions.

References

Arai, K., and Tagyo, K. (1985). “Determination of noncircular slip surfaces giving the minimum factor of safety in slope stability analysis.” *Soils. Found.*, 25(1), 43–51.

Baker, R., and Garber, M. (1978). “Theoretical analysis of the stability of slopes.” *Géotechnique*, 28(4), 395–411.

Bishop, A. W. (1955). “The use of the slip circle in the stability analysis of slopes.” *Géotechnique*, 5(1), 7–17.

Chen, Z., and Shao, C. (1988). “Evaluation of minimum factor of safety in slope stability analysis.” *Can. Geotech. J.*, 25(4), 735–748.

Cheng, Y. M. (2003). “Location of critical failure surface and some further studies on slope stability analysis.” *Comput. Geotech.*, 30(3), 255–267.

Cheng, Y. M., Li, L., Chi, S., and Wei, W. B. (2007a). “Particle swarm optimization algorithm for the location of the critical non-critical failure surface in two-dimensional slope stability analysis.” *Comput. Geotech.*, 34(2), 92–103.

Cheng, Y. M., Li, L., and Chi, S. C. (2007b). “Performance studies on six heuristic global optimization methods in the location of critical slip surface.” *Comput. Geotech.*, 34(6), 462–484.

Cheng, Y. M., Liang, L., Chi, S. C., and Wei, W. B. (2008). “Determination of the critical slip surface using article fish swarms algorithm.” *J. Geotech. Geoenviron. Eng.*, 10.1061/(ASCE)1090-0241(2008)134:2(244), 244–251.

Duncan, J. M., and Song, K. S. (1984). “SLOPE8R: A computer program for slope stability analysis with non-circular slip surfaces.” *Technical Rep.*, Dept. of Civil Engineering, Univ. of California, Berkeley, CA.

Duncan, J. M., and Wright, S. G. (2005). *Soil strength and slope stability*, Wiley, Hoboken, NJ.

Greco, V. R. (1996). “Efficient Monte Carlo technique for locating critical slip surface.” *J. Geotech. Eng.*, 10.1061/(ASCE)0733-9410(1996)122:7(517), 517–522.

Griffiths, D. V., and Lane, P. A. (1999). “Slope stability analysis by finite elements.” *Géotechnique*, 49(3), 387–403.

Jurado-Piña, R., and Jimenez, R. (2014). “A genetic algorithm for slope stability analyses with concave slip surfaces using custom operators.” *Eng. Optim.*,

Kahatadeniya, K. S., Nanakorn, P., and Neaupane, K. M. (2009). “Determination of the critical failure surface for slope stability analysis using ant colony optimization.” *Eng. Geol.*, 108(1–2), 133–141.

Li, K. S., and White, W. (1987). “Rapid evaluation of the critical slip surface in slope stability problems.” *Int. J. Numer. Anal. Meth. Geomech*, 11(5), 449–473.

Li, Y. C., Chen, Y. M., and Tony, L. T., et al. (2010). “An efficient approach for locating the critical slip surface in slope stability analyses using a real-coded genetic algorithm.” *Can. Geotech. J.*, 47(7), 806–820.

Lorenz, E. N. (1963). “Deterministic nonperiodic flow.” *J. Atmos. Sci.*, 20(2), 130–141.

Morgenstern, N. R., and Price, V. E. (1965). “Analysis of stability of general slip surfaces.” *Géotechnique*, 15(1), 79–93.

Malkawi, H. A. I., Hassan, W. F., and Saema, S. K. (2001). “Global search method for locating general slip surface using Monte Carlo techniques.” *J. Geotech. Geoenviron.*, 10.1061/(ASCE)1090-0241(2001)127:8(688), 688–698.

MATLAB. [Computer software]. MathWorks, Natick, MA.

Nguyen, V. U. (1985). “Determination of critical slope failure surfaces.” *J. Geotech. Eng.*, 10.1061/(ASCE)0733-9410(1985)111:2(238), 238–250.

Sarma, S. K., and Tan, D. (2006). “Determination of critical slip surface in slope analysis.” *Géotechnique*, 56(8), 539–550.

Schuster, R. L. (1996). “Socioeconomic significance of landslides.” *Landslides: Investigation and Mitigation, Special Rep. 247*, A. K. Turner and R. L. Schuster, eds., National Academy Press, Washington, DC, 12–35.

Solati, S., and Habibagahi, G. (2006). “A genetic approach for determining the generalized inter-slice forces and the critical non-critical slip surface.” *Iran. J. Sci. Technol. B.*, 30(1), 1–20.

Spencer, E. (1967). “Method of analysis of stability of embankments assuming parallel inter-slice forces.” *Géotechnique*, 17(1), 11–26.

Sun, J., Li, J., and Liu, Q. (2008). “Search for critical slip surface in slope stability analysis by spline-based GA method.” *J. Geotech. Geoenviron.*, 10.1061/(ASCE)1090-0241(2008)134:2(252), 252–256.

Tan, D., and Sarma, S. K. (2008). “Finite element verification of an enhanced limit equilibrium method for slope analysis.” *Géotechnique*, 58(6), 481–487.

Xu, L. M., Chen, J. P., Wang, Q., and Zhou, F. J. (2013). “Fuzzy C-means cluster analysis based on mutative scale chaos optimization algorithm for the grouping of discontinuity sets.” *Rock. Mech. Rock. Eng.*, 46(1), 189–198.

Yamagami, T., and Ueta, Y. (1988). “Search for noncircular slip surface by the Morgenstern-Price method.” *Proc., 6th Int. Conf. on Numerical Methods in Geomechanics*, A. A. Balkema, Rotterdam, 1335–1340.

Yang, D. X., Li, G., and Cheng, G. D. (2007). “On the efficiency of chaos optimization algorithms for global optimization.” *Chaos, Soliton Fractals*, 34(4), 1366–1375.

Zhang, T., Wang, H. W., and Wang, Z. C. (1999). “Mutative scale chaos optimization algorithm and its application.” *Control. Decis.*, 14(3), 285–288.

Zhang, X. (1999). “Slope stability analysis based on the rigid finite element method.” *Géotechnique*, 49(5), 585–593.

Zheng, H., Sun, G., and Liu, D. F. (2009a). “A critical procedure for searching critical slip surfaces of slope based on the strength reduction technique.” *Comput. Geotech.*, 36(1–2), 1–5.

Zheng, H., Zhou, C. B., and Liu, D. F. (2009b). “A robust solution procedure for the rigorous methods of slices.” *Soils Found.*, 49(4), 537–544.

Zolfaghari, A. R., Heath, A. C., and McCombie, P. F. (2005). “Simple genetic algorithm search for critical non-circular failure surface in slope stability analysis.” *Comput. Geotech.*, 32(3), 139–152.

Zou, J. Z., Williams, D. J., and Xiong, W. L. (1995). “Search for critical slip surfaces based on finite element method.” *Can. Geotech. J.*, 32(2), 233–246.